3. **Fundamentals of stability analyses**

3.1 **General**

Stability analyses for tunnels in rock are based on models representing the stress-strain behavior and the permeability of a jointed rock. The parameters which are used for these models are derived from the results of explorations, rock mechanics tests and monitoring and to a large extent from experience gained from projects in similar rock conditions (Wittke, 2000).

The effort invested in explorations and testing for machine-driving tunnels compared to tunnels, which are constructed using the conventional method, needs to be much higher, since mechanized tunneling is less adaptable to varying ground conditions. The application of TBM s in jointed rock mass requires very careful site investigations and stability analyses to keep risks small. The latter should be carried out with the aid of suitable numerical analysis methods (Section 3.5.1). The application of simplified models, which are treated in brief in section 3.5.2, as well as of classification systems is not sufficient and thus unsuitable for tunnels, particularly in case of machine-driven tunnels.

3.2 **Structural models**

Jointed rock mass consists of blocks of intact rock which are separated by discontinuities along which the consistency of the rock matrix is locally interrupted. The type, appearance and spatial orientation of grains in the intact rock is described by the grain fabric. The discontinuity fabric takes into account the orientation, location and appearance of discontinuities. The geometric setup of a rock mass therefore is usually quite complex, and can only be considered with idealized models in the design of structures in rock, e.g. in stability analysis for a tunnel. The grain and discontinuity fabric have a significant effect on the mechanical and hydraulic properties of a rock mass. As the basis for a quantitative description of the stress-strain behavior and of the water permeability of a rock mass, a model is used which superposes idealized grain and discontinuity fabrics. The result of such a superposition is referred to as structural model (Wittke, 1990).
A structural model classifies the grain fabrics of intact rocks appearing in nature with respect to the structure and texture of the crystals or grains. Most intact rocks can be related to a random, a planar or a linear grain fabric as represented by the models shown in Fig. 3.1.

Intact rocks with a random grain fabric are characterized by an isotropic behavior with regard to deformability and strength. In contrast to that, intact rocks with a planar or linear grain fabric may show a marked anisotropic behavior (see e.g. Pinto, 1970; Wittke, 1990).
With respect to tunneling, all intact rocks can be regarded as homogeneous because the individual particles are very small compared to the dimensions of a tunnel.

An idealized description of the discontinuity fabric by a structural model is based upon the principle of representing the discontinuities by parallel planes. Parameters describing orientation, extent, spacing, aperture and other properties of the discontinuities are evaluated with statistical methods.

As an example, Fig. 3.2 shows the structural models of three different types of rock mass. In Fig. 3.2a, a sandstone is represented, which can be described by a combination of a random grain fabric for the intact rock and a general three-dimensional system of discontinuities with three sets of discontinuities. The clay slate depicted in Fig. 3.2b also is separated by three sets of discontinuities, denoted with J1, J2 and S, but unlike the sandstone has a planar grain fabric. This is caused by the schistosity of the intact rock, which in this special case has the same orientation as the bedding. The basalt shown in Fig. 3.2c has a random grain fabric and is divided by three sets of discontinuities into column-like bodies.

Since the deformability, the strength and the permeability of a rock mass are strongly influenced by the discontinuities, an adequate description of the geometry of the discontinuity fabric is required for rock mechanics and stability analyses for a tunnel. Also the appearance of the discontinuities has to be characterized (Fig. 3.3). The surface properties of discontinuities are especially important with regard to the shear strength. In an idealizing manner, it can be distinguished between stepped, wavy and planar surfaces, which can each be rough, smooth or slickensided (Brown, 1981). Discontinuities are referred to as slickensided when their surfaces are smooth and even in at least one direction because of a relative shear displacement. Furthermore, discontinuities can be open or close and may have coatings or fillings (Fig. 3.3). Discontinuities do not always separate the rock completely, but are often interrupted by rock bridges (Fig. 3.3). The degree of separation can be described by the discontinuity spacing d and the planar degree of separation, which is the sum of the areas of the separated rock sections divided by the reference area (Packer, 1959).
Fig. 3.2: Different types of rock mass and corresponding structural models: a) sandstone; b) clay slate; c) basalt
Fig. 3.3: Characteristics of discontinuities

The three-dimensional orientation of a discontinuity is well-defined by two angles in a three-dimensional coordinate system (Fig. 3.4). The strike angle $\alpha$ is measured positively from the north axis clockwise towards the east axis until the intersection of the discontinuity and the horizontal plane through the center of a reference sphere. The angle $\beta$ between the horizontal plane and the line of dip of the discontinuity is referred to as the dip angle (Fig. 3.4).
The angles $\alpha$ and $\beta$ can be measured in situ with the aid of a geologist's compass (Fig. 3.4). The transformation of the orientation into the lower half of the reference sphere is also shown in Fig. 3.4. The normal of the discontinuity running through the center of the reference sphere intersects the sphere's surface at point P. P is referred to as the pole of the discontinuity. The orientation of a discontinuity is well-defined by the location of its pole.
To evaluate the orientations of discontinuities measured in situ, a representation of the poles in a polar equal-area net is suitable (Fig. 3.5). This net is created by an equal-area hemispheric projection of the lower half of the reference sphere onto a horizontal plane (Wittke, 1990). Among other things, Fig. 3.5 shows the pole P of the discontinuity represented in Fig. 3.4 with a strike angle of $\alpha = 12^\circ$ and a dip angle of $\beta = 67^\circ$.

![Fig. 3.5: Representation of poles in the polar equal-area net and evaluation of the measured data](image)

To group measured discontinuities into sets on the basis of the discontinuities' orientations, for each measured pair of values $\alpha/\beta$, one pole is entered into the polar equal-area net. It is then possible to determine lines of equal pole density with the aid of a method described in Wittke (1980). The areas lying between these lines are grouped into regions and assigned to a graded interval of pole density (Fig. 3.5). In this way, discontinuities are combined into sets according to the criterion of pole density. The point with maximum pole density in the polar equal-area net corresponds to the statistically most frequently measured orientation of the discontinuity set under consideration.
Fig. 3.5 shows an example in which it was possible to group the discontinuities into four different sets. The represented pole densities show that most frequently bedding-parallel discontinuities B and so-called transversal joints T were encountered. The frequencies of diagonal joints D and longitudinal joints L were of less importance.

The structural model represented in Fig. 3.6 is based on the orientations determined in accordance with Fig. 3.5, on measured values of spacing of the discontinuities and on information on the characteristics of the discontinuities (see Fig. 3.3). In the considered example, the rock mechanics properties are dominated by the bedding-parallel discontinuities B with mean orientations of $\alpha = 40^\circ$ and $\beta = 40^\circ$ (Fig. 3.5 and 3.6).

---

$\overline{d}_B = 1 \text{ m}$ : average spacing of bedding-parallel discontinuities
$\overline{t}_S = 3 \text{ cm}$ : average thickness of shear zones
$\overline{d}_S = 1.8 \text{ m}$ : average spacing of shear zones

---

Fig 3.6: Example of a structural model

The mean spacing of the widely persistent bedding-parallel discontinuities is $\overline{d}_B = 1 \text{ m}$. They are mostly filled. The mean thickness of these so-called shear zones amounts to $\overline{t}_S = 3 \text{ cm}$. The mean
spacing of these shear zones is $d_s = 1.8$ m. The transversal, longitudinal and diagonal joints, with orientations specified in Fig. 3.5, are normally limited by the bedding planes and only rarely extend through several layers of intact rock (Fig. 3.6). The mean spacing of the transversal joints ranges from 1 m to 3 m. In contrast, the mean spacing of the longitudinal and diagonal joints amounts to several meters.

Based on a structural model, the stress-strain behavior and the water permeability of a rock mass can be described in an optimal way. A rock mechanical investigation should therefore always start with the development of a structural model of the rock mass.

![Fig. 3.7: Examples of grain and discontinuity fabrics and corresponding rock mechanical models for elastic behavior (Wittke, 1990)](image-url)

In Fig. 3.7, some examples of grain and discontinuity fabrics and corresponding rock mechanical models for elastic behavior, which are discussed below, are shown.
3.3 Models for stress-strain behavior of jointed rock

3.3.1 On the stress-strain behavior of jointed rock

Discontinuities influence the rock mechanics properties, in particular because of the fact that shear strength in parallel with and tensile strength perpendicularly to the discontinuities are lower than the strength of the intact rock. Furthermore, the deformability of a rock mass perpendicularly to the discontinuities is normally higher than that of the intact rock. Because of this influence, the rock mass in many cases shows an anisotropic behavior under mechanical loading.

For the rock mass, a linear elastic-viscoplastic stress-strain behavior is assumed. Here, it is assumed that stresses below strength only lead to elastic strains, which are independent of time and proportional to the stresses. If strength is exceeded, non-elastic irreversible strains will occur. These can either indefinitely increase and lead to complete failure of the loaded rock mass area, or they remain limited, if due to stress redistribution within the loaded rock mass area a new state of equilibrium can be reached. Experience shows that these irreversible strains and the corresponding stress redistributions depend on time. Therefore, the theory of viscoplasticity is used for their description. If in the surrounding of a tunnel the strength of the intact rock is exceeded to a large extent, leading to great time dependent displacements which cannot be avoided by the immediate installation of a support, the term squeezing rock is used in tunneling.

3.3.2 Elastic behavior

Intact rock

The elastic behavior of intact rock with planar grain fabric (schistosity or bedding) can be assumed as transversely isotropic. It can be described by means of five elastic constants, which are independent of each other (Fig. 3.8). Usually, the Young's moduli $E_1$ and $E_2$ are used for loading in parallel with and perpendicular to the isotropic plane (schistosity and bedding, respectively) and the shear modulus $G_2$ is used for shear loading in the isotropic plane. Further, the Poisson's ratios $\nu_1$ and $\nu_2$ determine the elastic behavior of such a material (Wittke, 1990).
The elastic behavior of intact rock with random grain fabric can be assumed as isotropic. The elastic behavior can then be described by two elastic constants, the Young's modulus \( E \) and the Poisson's ratio \( \nu \). Because of the relationships

\[
E_1 = E_2 = E, \\
\nu_1 = \nu_2 = \nu, \\
G_2 = \frac{E}{2(1 + \nu)},
\]

the isotropic elastic behavior is a special case of the transversely isotropic elastic behavior.

In Fig. 3.8, the elastic constants for the case of transversely isotropic elastic behavior are defined by means of normal stresses \( \sigma \) and shear stresses \( \tau \), respectively, as well as the corresponding normal strains \( \varepsilon \) and shear strains \( \gamma \), respectively, in a three-dimensional element. The utilized cartesian coordinate system \((x', y', z')\) is related to the isotropic plane. The \( x' \)- and \( y' \)-axes lie in the isotropic plane and the \( z' \)-axis coincides with the direction of the highest deformability. In Fig. 3.8, the elastic constants \( v_3 \) and \( G_1 \), which are dependent on \( E_1, E_2, v_1 \) and \( v_2 \), are also defined.

In the coordinate system \((x', y', z')\) related to the isotropic plane, the stress-strain relationship for the case of transverse isotropy can be expressed as follows:

\[
\{\sigma'\} = [D'] \cdot \{\varepsilon'\}
\]

with

\[
\{\sigma'\} = (\sigma_{x'}, \sigma_{y'}, \sigma_{z'}, \tau_{x'y'}, \tau_{x'z'}, \tau_{y'z'})^T, \\
\{\varepsilon'\} = (\varepsilon_{x'}, \varepsilon_{y'}, \varepsilon_{z'}, \gamma_{x'y'}, \gamma_{x'z'}, \gamma_{y'z'})^T
\]

\[
[D'] = \begin{bmatrix}
E_1 \frac{1 - n \nu_2^2}{(1 + \nu_1)m} & E_1 \frac{\nu_1 + n \nu_2^2}{(1 + \nu_1)m} & E_1 \frac{\nu_2}{m} & 0 & 0 & 0 \\
E_1 \frac{\nu_1 + n \nu_2^2}{(1 + \nu_1)m} & E_1 \frac{1 - n \nu_2^2}{(1 + \nu_1)m} & E_1 \frac{\nu_2}{m} & 0 & 0 & 0 \\
E_1 \frac{\nu_2}{m} & E_1 \frac{\nu_2}{m} & E_2 \frac{1 - \nu_1}{m} & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{E_1}{2(1 + \nu_1)} & 0 & 0 \\
0 & 0 & 0 & 0 & G_2 & 0 \\
0 & 0 & 0 & 0 & 0 & G_2
\end{bmatrix}
\]
In (3.1), the abbreviations $n = \frac{E_1}{E_2}$ and $m = 1 - \nu_1 - 2\nu_2^2$ have been used.

Fig. 3.8: Definition of elastic constants for an intact rock with transversely isotropic elastic behavior (Wittke, 1990)
The inverse relation of (3.1) is given by:

\[ \{\varepsilon\}' = [D']^{-1} \cdot \{\sigma\}' \quad (3.2) \]

with

\[
[D']^{-1} = \begin{bmatrix}
\frac{1}{E_1} & -\frac{v_1}{E_1} & -\frac{v_2}{E_1} & 0 & 0 & 0 \\
-\frac{v_1}{E_1} & \frac{1}{E_1} & -\frac{v_2}{E_1} & 0 & 0 & 0 \\
-\frac{v_2}{E_1} & -\frac{v_2}{E_1} & \frac{1}{E_1} & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{2(1 + v_1)}{E_1} & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{G_2} & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{G_2}
\end{bmatrix}
\]

When carrying out stability analyses, a global coordinate system 
\((x, y, z)\) related to the engineering structure’s overall geometry
is used, which normally does not coincide with the coordinate sys-
tem \((x', y', z')\) introduced above. Linking of the two coordinate
systems can be achieved with the aid of two angles. These describe
the direction of the contour line (strike \(\alpha\)) and the inclination
of the line of dip (dip \(\beta\)) of the isotropic plane in relation to
the global coordinate system. The definition of the angles \(\alpha\) and \(\beta\)
is given in Fig. 3.9 (see also Fig. 3.4).

Fig. 3.9: Relationship between the global coordinate system
and the coordinate system related to the orientation
of a discontinuity, schistosity or bedding
Stresses \( \{ \sigma' \} \) and strains \( \{ \varepsilon' \} \) can be computed from the stresses \( \{ \sigma \} \) and strains \( \{ \varepsilon \} \) in the global coordinate system with the aid of transformation matrices \([T]\) and \([T^*]\), respectively:

\[
\{ \sigma' \} = [T] \cdot \{ \sigma \},
\]

(3.3)

\[
\{ \varepsilon' \} = [T^*] \cdot \{ \varepsilon \}.
\]

(3.4)

The dependency of \([T]\) and \([T^*]\) from the angles \( \alpha \) and \( \beta \) is given by the following equations:

\[
[T] =
\begin{bmatrix}
  l_1^2 & m_1^2 & 0 & 2l_1m_1 & 0 & 0 \\
  l_2^2 & m_2^2 & n_2^2 & 2l_2m_2 & 2m_2n_2 & 2n_2l_2 \\
  l_3^2 & m_3^2 & n_3^2 & 2l_3m_3 & 2m_3n_3 & 2n_3l_3 \\
  l_1l_2 & m_1m_2 & 0 & l_1m_2 + l_2m_1 & m_1n_2 & n_2l_1 \\
  l_2l_3 & m_2m_3 & n_2n_3 & l_2m_3 + l_3m_2 & m_2n_3 + m_3n_2 & n_3l_2 + n_2l_2 \\
  l_3l_1 & m_3m_1 & 0 & l_3m_1 + l_1m_3 & m_3n_1 & n_1l_3
\end{bmatrix}
\]

(3.5)

\[
[T^*] =
\begin{bmatrix}
  l_1^2 & m_1^2 & 0 & l_1m_1 & 0 & 0 \\
  l_2^2 & m_2^2 & n_2^2 & l_2m_2 & m_2n_2 & n_2l_2 \\
  l_3^2 & m_3^2 & n_3^2 & l_3m_3 & m_3n_3 & n_3l_3 \\
  2l_1l_2 & 2m_1m_2 & 0 & l_1m_2 + l_2m_1 & m_1n_2 & n_2l_1 \\
  2l_2l_3 & 2m_2m_3 & 2n_2n_3 & l_2m_3 + l_3m_2 & m_2n_3 + m_3n_2 & n_3l_2 + n_2l_2 \\
  2l_3l_1 & 2m_3m_1 & 0 & l_3m_1 + l_1m_3 & m_3n_1 & n_1l_3
\end{bmatrix}
\]

(3.6)

with

\[
\begin{align*}
  l_1 &= \sin \alpha, & m_1 &= \cos \alpha, \\
  l_2 &= \cos \beta \cos \alpha, & m_2 &= -\cos \beta \sin \alpha, & n_2 &= -\sin \beta, \\
  l_3 &= -\sin \beta \cos \alpha, & m_3 &= \sin \beta \sin \alpha, & n_3 &= -\cos \beta.
\end{align*}
\]

Insertion of (3.3) and (3.4) in (3.1) yields:

\[
[T] \cdot \{ \sigma \} = [D'] \cdot [T^*] \cdot \{ \varepsilon \}.
\]

Multiplication by \([T]^{-1}\) gives

\[
\{ \sigma \} = [T]^{-1} \cdot [D'] \cdot [T^*] \cdot \{ \varepsilon \}.
\]
The calculation of $[T]^{-1}$, which is not explicitly presented here, leads to the relationship

$$[T]^{-1} = [T']^T. \quad (3.7)$$

Thus (3.1), can be replaced by

$$\{\sigma\} = [D] \cdot \{\varepsilon\} \quad (3.8)$$

with

$$[D] = [T']^T \cdot [D'] \cdot [T'].$$ 

Equation (3.8) describes the relation between stresses and strains in an arbitrary global coordinate system, which is linked to the coordinate system $(x', y', z')$ related to the isotropic plane in the way represented in Fig. 3.9.

The inverse relation of (3.8) can be achieved by adequate rearrangements:

$$\{\varepsilon\} = [T]^{-1} \cdot [D']^{-1} \cdot [T] \cdot \{\sigma\}. \quad (3.9)$$

For the special case of isotropic intact rocks with random grain fabric, the $[D]$-matrix is independent of the orientation of the coordinate system and coincides with the $[D']$-matrix.

Not all intact rocks can be described by isotropic or transversely isotropic behavior in the elastic domain. A generalization of the transverse isotropy is denoted as orthotropy. In this case, nine independent elastic constants are required to describe the elastic behavior. The general anisotropic body with no symmetry at all can be described by 21 elastic constants (Lekhnitskii, 1963). The efforts for the determination of more than five elastic constants by means of laboratory tests on rock samples, however, is usually unjustifiably high. Therefore, this is done only in exceptional cases.

**Rock mass**

The "homogenous model" suggested by Wittke (1990) usually is taken as a basis for the description of the stress-strain behavior of a jointed rock mass. This model allows to consider sets of disconti-
nuities in stability analyses with relatively low effort, taking into account the anisotropy of deformability and strength caused by the grain and discontinuity's fabric.

Fig. 3.10: Idealization of soil and jointed rock mass using a homogeneous substitute material (Wittke, 1990): a), c) Discrete model; b, d) homogeneous model
In the homogeneous model, average stresses and strains are used, which include the deformations of the intact rock as well as the deformations resulting from displacements on discontinuities. The elastic constants of the rock mass in the homogeneous model are related to the average stresses and strains and therefore differ from the elastic constants of the intact rock. This particularly applies to the Young's moduli, which in general will be smaller than those of the intact rock due to the higher deformability of the discontinuities.

In Fig. 3.10, the idealization of a soil and a rock mass containing one set of discontinuities for the case of a one-dimensional loading is illustrated (Wittke, 1990).

![Diagram](image)

Fig. 3.11: Applicability of the homogeneous model to a jointed rock mass in tunneling (Wittke, 2000)

The application of the homogenous model, however, leads to reliable results only when the dimensions of the tunnel are large in comparison with those of the discontinuities' spacing and of the blocks bonded by the discontinuities, respectively (Fig. 3.11). Therefore, master joints and fault zones, for which this assumption normally is not fulfilled, are considered or simulated discretely.
A discrete modeling of discontinuities in statical analyses is often carried out in a simplified manner, assuming a linear elastic-viscoplastic stress-strain behavior of the discontinuities and considering an ideal discontinuity thickness \( t_\sigma \). The elastic behavior below strength is described by the linear stress strain relationship explained above for intact rocks. The elastic constants describe the mean deformability of the area specified by the discontinuity's thickness \( t_\sigma \). For discontinuities filled with soil, for example, the discontinuity's thickness \( t_\sigma \) corresponds to the thickness of the filling of the discontinuity, and the deformability of the discontinuity is determined by the elastic constants of the soil.

### 3.3.3 Strength, failure and post-failure behavior of intact rock

The isotropic shear strength of intact rock with random grain fabric and of intact rock with planar grain fabric in directions other than that of the fabric's plane is described by the Mohr-Coulomb failure criterion:

\[
\tau = \sigma \cdot \tan \varphi_{IR} + c_{IR}. \tag{3.10}
\]

\( \sigma \) is the normal stress acting on a shear plane and \( \tau \) the corresponding shear stress in state of failure, \( c_{IR} \) is the cohesion and \( \varphi_{IR} \) the angle of internal friction. Because of the isotropy of strength, (3.10) represents the envelope of the Mohr's circles of stress in state of failure idealized to a straight line in the \( \tau-\sigma \)-diagram (Fig. 3.12a). The Mohr's circles of stress are defined by the maximum and minimum principal stresses \( \sigma_1 \) and \( \sigma_3 \). Using the relationships valid for a Mohr's circle of stress, the failure line described by (3.10) can also be expressed in terms of the principal normal stresses:

\[
\sigma_1 = \frac{1 + \sin \varphi_{IR}}{1 - \sin \varphi_{IR}} \cdot \sigma_3 \cdot \frac{2 \cdot \cos \varphi_{IR}}{1 - \sin \varphi_{IR}}. \tag{3.11}
\]

This equation describes a straight line in the \( \sigma_1-\sigma_3 \)-diagram (Fig. 3.12b). While the inclination \( \beta \) of this straight line is a function of the angle of internal friction \( \varphi_{IR} \), the point of intersection on the \( \sigma_1 \)-axis represents the unconfined compressive strength \( \sigma_{IR} \).
Fig. 3.12: Mohr-Coulomb failure criterion and tension cut-off criterion for isotropic strength behavior: 
a) $\tau-\sigma$-diagram; b) $\sigma_1-\sigma_3$-diagram

According to the sign convention common in Geotechnics, compressive stresses are specified as positive and tensile stresses as negative in Fig. 3.12.

The failure criterion explained above can also be formulated with a function $F$:

$$F = \tau - \sigma \cdot \tan \varphi_{IR} - c_{IR} \quad (3.12)$$

or

$$F = \frac{1}{2} \cdot \sigma_1 \cdot (1 - \sin \varphi_{IR}) - \frac{1}{2} \cdot \sigma_3 \cdot (1 + \sin \varphi_{IR}) - c_{IR} \cdot \cos \varphi_{IR}. \quad (3.13)$$

At $F > 0$, the state of stress is situated above the failure line (Fig. 3.12) and the shear strength of the intact rock has been exceeded. For states of stress located on the failure line, the function $F$ is zero. At $F < 0$, the state of stress lies below the failure line and the behavior of the intact rock is elastic. The function $F$ is referred to as flow function.

Regarding the tensile failure, it is assumed that cracking of intact rock occurs perpendicularly to the direction of the principal normal stress which exceeds the tensile strength $\sigma_{tIR}$. This crite-
rion can be represented in the $\sigma_1-\sigma_3$-diagram as well as in the $\tau-\sigma$-diagram as the vertical straight line (Fig. 3.12)

$$\sigma_3 = -\sigma_{\text{TIR}}.$$ \hfill (3.14)

On the right-hand side of this straight line, the above described failure criterion for shear failure applies. States of stress corresponding to a point on or left of this straight line lead to a tensile failure. This combination of failure criteria for tensile and shear failure is referred to as "tension cut-off" criterion. Formulating the criterion for tensile failure with a flow function $F$, it can be expressed as:

$$F = -\sigma_3 - \sigma_{\text{TIR}}.$$ \hfill (3.15)

The Mohr-Coulomb failure criterion is also applied to describe the shear strength of the fabric's plane in intact rocks with planar grain fabric. It is formulated for the resultant shear stress $\tau_{\text{res}}$ in this plane and the corresponding normal stress $\sigma_n$ (Fig. 3.13):

$$\tau_{\text{res}} = \sigma_n \cdot \tan \varphi_s + c_s$$ \hfill (3.16)

or

$$F = \tau_{\text{res}} - \sigma_n \cdot \tan \varphi_s - c_s.$$ \hfill (3.17)

Fig. 3.13: Criteria for shear and tensile failure on a fabric's plane or discontinuity $S$ (Wittke, 1990)
For a tensile failure perpendicular to the fabric's plane of the planar grain fabric, the following failure criterion is used:

\[ \sigma_n = - \sigma_{ts} \]  

(3.18)

or

\[ F = - \sigma_n - \sigma_{ts}. \]  

(3.19)

The combination of both failure criteria for shear and tension is illustrated in Fig. 3.13. The normal stress \( \sigma_n \) acting on the fabric's plane and the resultant shear stress \( \tau_{res} \) acting in parallel with this plane can in general be determined from the corresponding state of stress \( \{\sigma\} \) described in the global x-y-z coordinate system (Wittke, 1990). The reduction in strength on the fabric's plane is taken into account by the selection of adequate shear parameters \( \varphi_s \) and \( c_s \) as well as for the tensile strength \( \sigma_{ts} \).

The anisotropic strength of an intact rock with a planar grain fabric for shear and tensile loadings can thus be described by a combination of the criteria specified by the equations (3.13) and (3.15) for directions deviating from the fabric's plane direction, as well as (3.17) and (3.19) for the direction of the fabric plane. For a given state of stress it must be checked whether one or more and, if so, which of these failure criteria are violated.

**Residual strength**

As a rule, intact rock can still transfer shear stresses to a limited extent after the shear strength has been exceeded. This takes mainly place via friction. The residual angle of internal friction \( \varphi_{IR}^* \), however, is not necessarily equal to the angle \( \varphi_{IR} \). In most cases, the cohesion is clearly reduced after the shear strength has been exceeded, the residual value being referred to as \( c_{IR}^* \). In case of a tensile failure, the residual tensile strength \( \sigma_{tIR}^* \) is normally assumed to zero. In the following, the failure criterion is denoted as \( F' \), if the strength parameters \( c_{IR}, \varphi_{IR} \) and \( \sigma_{tIR} \) are replaced by the residual values \( c_{IR}^*, \varphi_{IR}^* \) and \( \sigma_{tIR}^* \).

**Viscoplastic behavior after exceeding of strength**

After exceeding the strength, irreversible strains occur. These often show a distinctive dependence on time and are referred to as
viscoplastic strains. For the analysis of these strains, the theory of viscoplasticity is applied. The viscoplastic strain rates \( \dot{\epsilon}^{vp} \) related to time are used. According to Perzyna (1965), these are defined using the derivative of the scalar function \( Q_{IR} \), the so-called plastic potential, with respect to the components of the stress vector. For an exceeding of strength in isotropic intact rock, for example, the following equation is used:

\[
\{ \dot{\epsilon}^{vp} \} = \frac{1}{\eta_{IR}} \cdot F_{IR} \cdot \frac{\partial Q_{IR}}{\partial Q}
\]

\( \eta_{IR} \) is referred to as viscosity of the intact rock with regard to viscoplastic strains. \( F_{IR} \) and \( F_{IR}^* \) represent the failure criterion of the intact rock, with the values of the strength parameters at failure \( (F_{IR}) \) and the residual strength parameters \( (F_{IR}^*) \), respectively. \( Q_{IR} \) is the plastic potential of the intact rock. The checking whether viscoplastic strains occur is at first done using the failure criterion \( F_{IR} \). After exceeding of strength \( (F_{IR} > 0) \), \( F_{IR}^* \) serves as criterion to check whether the viscoplastic strains go on increasing or come to a halt. This means that it is assumed that after exceeding the failure strength \( (F_{IR} > 0) \) an immediate softening independent of the magnitude of the viscoplastic strains takes place and the strength is then reduced to the residual strength.

Equation (3.20) is known as flow rule. The plastic potential \( Q_{IR} \) is defined analogously to flow function \( F_{IR} \). For a shear failure in isotropic intact rock, \( Q_{IR} \) has the following form:

\[
Q_{IR} = \frac{1}{2} \cdot \sigma_1 \cdot (1 - \sin \psi_{IR}) - \frac{1}{2} \cdot \sigma_3 \cdot (1 + \sin \psi_{IR}) - c_q \cdot \cos \psi_{IR} \, . \quad (3.21)
\]

The angle of dilatancy \( \psi_{IR} \) specifies the volume increase or loosening of the intact rock, respectively, when the shear strength is exceeded. At \( \psi_{IR} = \phi_{IR} \), \( Q_{IR} \) is identical to \( F_{IR} \). In this case, we are talking about an associated flow rule. Applying an associated flow rule, the viscoplastic volumetric strain is often overestimated. Therefore, usually an angle of dilatancy is chosen which is smaller than \( \phi_{IR} \). In this case, \( Q_{IR} \neq F_{IR} \) and we are talking about a non-associated flow rule. For \( \psi_{IR} = 0 \), no volume increase of the intact rock occurs after the shear strength is exceeded. The angle
of dilatancy has a marked influence on viscoplastic strains which occur when the strength is exceeded.

The tensile failure in isotropic intact rock is described by an associated flow rule:

\[ Q_{IR} = F_{IR} = -\sigma_3 - \sigma_{tIR}. \] \hspace{1cm} (3.22)

The viscoplastic strains can be obtained from the strain rates by integration with respect to time:

\[ \{\varepsilon^{vp}(t)\} = \int_{0}^{t} \{\dot{\varepsilon}^{vp}\} \cdot dt. \] \hspace{1cm} (3.23)

Due to stress redistributions in the system caused by viscoplastic strains, function \( F_{IR} * \) is generally dependent on time so that the integration in (3.23) can normally be carried out by approximation only. In numerical analyses, this is done iteratively in a time-step-analysis starting at \( t = 0 \) and calculating the increase of viscoplastic strains and the change in stresses in time steps \( \Delta t \) for times \( t = \Delta t, 2\Delta t, 3\Delta t \) etc. It is assumed that \( \{\dot{\varepsilon}^{vp}\} \) remains constant in each time step. The integral (3.23) is then replaced by the summation:

\[ \{\varepsilon^{vp}(t = m \cdot \Delta t)\} = \sum_{n=0}^{m} \{\dot{\varepsilon}^{vp}(t = n \cdot \Delta t)\} \cdot \Delta t. \] \hspace{1cm} (3.24)

This time-history analysis remains numerically stable and leads to reasonable results, only if the time steps are selected adequately small (Cormeau, 1975).

The knowledge of the viscosity \( \eta_{IR} \) is not necessary in order to calculate strains for the state of equilibrium, which is frequently of interest in practical applications. If \( \eta_{IR} \) is not known, it can be replaced by an arbitrarily selected value \( \bar{\eta}_{IR} \). In such cases, \( \{\varepsilon^{vp}(t)\} \) is computed according to (3.24) using a fictitious time step \( \Delta \bar{E} \). If for the investigated system a state of equilibrium is possible, a time-history analysis with \( \bar{\eta}_{IR} \) and \( \Delta \bar{E} \) leads to the same strains as a corresponding analysis with \( \eta_{G} \) and \( \Delta t \), as far as the analysis is numerically stable. If no state of equilibrium is possible, the time-history analysis with \( \bar{\eta}_{IR} \) and \( \Delta \bar{E} \) as well as the one with \( \eta_{IR} \) and \( \Delta t \) lead to strains increasing above all limits. A time-history analysis with fictitious values \( \bar{\eta}_{IR} \) and \( \Delta \bar{E} \),
which are used to compute the state of equilibrium, is referred to as viscoplastic iterative analysis.

For the analysis of stresses, the total strains are subdivided into the elastic and the viscoplastic components:

\[ \{\varepsilon\} = \{\varepsilon^e\} + \{\varepsilon^v\}. \quad (3.25) \]

Only the elastic strains lead to stresses for which the following applies according to equation (3.8):

\[ \{\sigma\} = [D] \cdot \{\varepsilon^e\}. \quad (3.26) \]

The stresses are thus calculated from the total strains as follows:

\[ \{\sigma\} = [D] \cdot \left(\{\varepsilon\} - \{\varepsilon^v\}\right). \quad (3.27) \]

Equation (3.27) represents the stress-strain relationship for an elastic-viscoplastic material.

The viscoplastic strain rates in an intact rock with a planar grain fabric, which are caused by an exceeding of strength in other directions than that of the fabric's plane, can be calculated analogously. For an exceeding of strength in the fabric's plane, the following flow rule applies:

\[ \dot{\varepsilon}^v = \frac{\eta_s}{s} \cdot \frac{\partial Q_s}{\partial \sigma} \quad \text{for } F_s > 0 \quad \text{and } F_s^* > 0, \quad \text{resp.} \quad (3.28) \]

\( F_s \) and \( F_s^* \) correspond to the failure criterion for the fabric's plane with values for the failure strength parameters and the residual strength parameters, respectively. The viscosity \( \eta_s \) does not necessarily coincide with \( \eta_{IR} \). For the case of shear failure, the plastic potential \( Q_s \) has the following form:

\[ Q_s = \tau_{res} - \sigma_n \cdot \tan \psi_s - \sigma_s. \quad (3.29) \]

Depending on the value of the angle of dilatancy \( \psi_s \), the flow rule for shear failure is either associated (\( \psi_s = \psi_s \)) or non-associated (\( \psi_s \neq \psi_s \)). For \( \psi_s > 0 \), a shear failure on the fabric's plane results in shear strains as well as in a strain component normal to the fabric's plane, which leads to an increase of volume or, in
case this strain is impeded, to an increase in normal stress. This strain component increases with increasing $\psi_s$.

For a tensile failure perpendicular to the fabric's plane, $Q_s$ is assumed to be identical with $F_s$:

$$Q_s = F_s = - \sigma_n - \sigma_{ts}. \quad (3.30)$$

If the intact rock strength is exceeded in the fabric's plane as well as in directions other than that of the fabric's plane, the strain rates according to (3.20) and (3.28) are to be superposed:

$$\{ \dot{\varepsilon}^{vp} \} = \frac{1}{\eta_{ir}} \cdot F_{ir} \cdot \frac{\partial Q_{ir}}{\partial (\sigma)} + \frac{1}{\eta_s} \cdot F_s \cdot \frac{\partial Q_s}{\partial (\sigma)}. \quad (3.31)$$

The computation of strains and stresses is then carried out with $\{ \dot{\varepsilon}^{vp} \}$ according to (3.31) in the same manner as described for isotropic intact rock (equations (3.23) to (3.27)).

### 3.3.4 Failure and post-failure behavior of discontinuities

The stress-displacement behavior of discontinuities will not be discussed in detail here. Erichsen (1987) and Hartmann (1995) have described this behavior in detail and gave extensive compilations of models for the description.

The description of the shear strength on discontinuities is usually also based on the Mohr-Coulomb failure criterion, whereas the tensile strength is normally assumed to be zero. The failure criteria for discontinuities are formulated analogously to those for an intact rock with planar grain fabric, with the resultant shear stress $\tau_{res}$ acting in and the normal stress $\sigma_n$ acting perpendicularly to the discontinuity (equations (3.17) and (3.19)).

The irreversible strain rates resulting from the exceeding of the discontinuity's strength are calculated with the following flow rule under consideration of the above mentioned ideal discontinuity's thickness $t_d$:

$$\{ \dot{\varepsilon}^{vp} \} = \frac{1}{\eta_d} \cdot F_d \cdot \frac{\partial Q_d}{\partial (\sigma)} \text{ for } F_d > 0 \text{ and } F_d' > 0. \quad (3.32)$$
The irreversible relative displacements of the discontinuity's walls opposite to each other can be calculated from \( \{ \varepsilon^{p} \} \) and \( t_{D} \). In the case of a shear failure, the plastic potential \( Q_{D} \) has the same form as the flow function \( F_{D} \), as in the case of shear failure on the fabric's plane of an intact rock with a planar grain fabric. The angle of internal friction \( \phi_{D} \) is then replaced by an angle of dilatancy \( \psi_{D} \). In case of tensile failure perpendicular to the discontinuity, \( Q_{D} = F_{D} \) applies.

In the case of filled discontinuities, the strength parameters, in particular those describing the shear strength, are determined basically by the strength parameters of the filling. In this case, the failure criteria and flow rules for the discontinuities must be formulated using the strength parameters of the filling \( \phi_{F}, c_{F}, \phi_{F}^{*}, c_{F}^{*} \) and \( \sigma_{tF} \).

### 3.3.5 Failure and post-failure behavior of a rock mass

The strength of a jointed rock mass results from the superposition of the intact rock strength and the strengths of different sets of discontinuities. In the homogenous model, it is assumed that a discontinuity with the mean orientation of each set of discontinuities exists in each point of the continuum. With this assumption, inhomogeneities of stress distribution caused by discontinuities are not taken into account. Therefore, the failure criteria formulated for intact rock and discontinuities must be expressed, in terms of the homogeneous model, as a function of the mean stresses computed with this model. The smaller the spacing of discontinuities of individual sets in comparison with the considered volume, the closer the actual conditions can be approached with the homogenous model (see Fig. 3.11).

Using the homogeneous model, the flow rule to calculate the visco-plastic strain rates of a rock mass with \( n \) sets of discontinuities (\( D_{1} \) to \( D_{n} \)) and a schistosity (\( s \)) is obtained from the sum of the strain rates for the intact rock, for the schistosity and for the individual sets of discontinuities:

\[
\{ \varepsilon^{p} \} = \frac{1}{\eta_{IR}} \cdot F_{IR} \cdot \frac{\partial Q_{IR}}{\partial [\sigma]} + \frac{1}{\eta_{s}} \cdot F_{s} \cdot \frac{\partial Q_{s}}{\partial [\sigma]} + \sum_{i=1}^{n} \frac{1}{\eta_{Di}} \cdot F_{Di} \cdot \frac{\partial Q_{Di}}{\partial [\sigma]}.
\]

This presupposes that the strength of the intact rock, the strength on the schistosity and the strengths on the sets of dis-
continuities are all exceeded. Otherwise, the corresponding term is omitted from the summation.

According to the idea on which the homogeneous model is based, viscoplastic strains will be computed for each point, if the strength on a set of discontinuities is exceeded. This results in the calculation of normal strains and shear strains related to a finite rock volume, whereas the use of a discrete model results in the relative displacements of the individual discontinuity's walls. Provided that the viscoplastic components of deformation caused by violation of the failure criteria for a discontinuity set $D_i$ are not or only to a small degree dependent on time, the viscosity $\eta_{D_i}$ practically is of no physical importance. In such a case, a clearly smaller value in comparison with the viscosities of intact rock and other sets of discontinuities must be chosen for $\eta_{D_i}$.

It should be pointed out that the discrete and homogeneous model can be coupled by discretely modeling individual large joints and faults of particular importance, while the influence of the discontinuity sets on strength and deformability is taken into account using the homogenous model.

The consideration of discontinuities as well as fabric planes in the description of the stress-strain behavior of a rock mass is of great importance, as illustrated by the following example.

In this example, the shear strength of a rock mass with two mutually perpendicular sets of discontinuities is considered (Wittke, 2000; Wittke and Wittke-Gattermann, 2005). The shear strength of the rock mass is obtained from a superposition of the intact rock's shear strength and of the shear strength in parallel with both sets of discontinuities. The latter are assumed at every point in accordance with the homogeneous model. As a consequence, a directional dependence of strength arises, which will be explained in the following with the aid of the rock block illustrated in Fig. 3.14. The rock block is loaded by the principal normal stresses $\sigma_1$ and $\sigma_3$. It consists of intact rock with isotropic strength and is separated by two sets of discontinuities D1 and D2 with reduced shear strength and differing inclinations, which are defined by the angles of dip $\beta_1$ and $\beta_2 = 90^\circ - \beta_1$ varying between $0^\circ$ and $90^\circ$. The intact rock's shear strength or the shear strengths on the discontinuity sets are exceeded depending on $\sigma_3$. 
The normal stress $\sigma_n$ and the resultant shear stress $\tau_{\text{res}}$ on the discontinuities D1 and D2 can be derived as a function of $\sigma_1$, $\sigma_3$ and the angle of dip $\beta$ from the relationships valid for Mohr’s circle of stress. The normal stress $\sigma_n$ perpendicular to a plane which is inclined at an angle $\beta$, and the resultant shear stress on this plane $\tau_{\text{res}}$ are obtained, when in the center of Mohr’s circle the radius $(\sigma_1 - \sigma_3)/2$ is plotted under an angle of $2\beta$ (Fig. 3.15). In the area of the red colored right-angled triangle represented in Fig. 3.15, $\sigma_n$ and $\tau_{\text{res}}$ can be specified as follows:

$$\sigma_n(\beta) = \frac{\sigma_1 + \sigma_3}{2} - \frac{\sigma_1 - \sigma_3}{2} \cos(180^\circ - 2\beta)$$
$$= \frac{\sigma_1 + \sigma_3}{2} + \frac{\sigma_1 - \sigma_3}{2} \cos 2\beta,$$  \hspace{1cm} (3.34)

$$\tau_{\text{res}}(\beta) = \frac{\sigma_1 - \sigma_3}{2} \sin(180^\circ - 2\beta)$$
$$= \frac{\sigma_1 - \sigma_3}{2} \sin 2\beta.$$  \hspace{1cm} (3.35)

Fig. 3.14: Principal normal stress $\sigma_1$ at failure for a rock mass with two mutually perpendicular sets of discontinuities
Inserting (3.34) and (3.35) into the Mohr-Coulomb failure criterion (3.17) for a discontinuity D and solving for \( \sigma_1 \), the following equation is obtained:

\[
\sigma_1 = \frac{(\sin 2\beta \cos \phi_d + \sin \phi_d - \cos 2\beta \sin \phi_d)\sigma_3 + 2c_d \cos \phi_d}{\sin 2\beta \cos \phi_d - \sin \phi_d - \cos 2\beta \sin \phi_d}
\]  

(3.36)

or, after application of the appropriate trigonometric theorems,

\[
\sigma_1 = \frac{[\sin(2\beta - \phi_d) + \sin \phi_d] \sigma_3 + 2c_d \cos \phi_d}{\sin(2\beta - \phi_d) - \sin \phi_d}.
\]  

(3.36a)

**Fig. 3.15:** Determining the normal and shear stress on a discontinuity using Mohr's circle of stress

The maximum principal stress \( \sigma_1 \), which leads to a shear failure of the intact rock or on the discontinuities for the angles of dip \( \beta_1 \) and \( \beta_2 \) respectively, is obtained from the following equation as a function of the shear parameters of the intact rock (\( c_{IR} \) and \( \varphi_{IR} \)) and of the two sets of discontinuities (\( c_{D1} \), \( \varphi_{D1} \), \( c_{D2} \) and \( \varphi_{D2} \)):

\[
\sigma_1 = \text{Minimum} \left\{ \frac{[\sin(2\beta_1 - \varphi_{D1}) + \sin \varphi_{D1}] \sigma_3 + 2c_{D1} \cos \varphi_{D1}}{\sin(2\beta_1 - \varphi_{D1}) - \sin \varphi_{D1}} \right\}, \quad (3.37)
\]

\[
\sigma_1 = \left\{ \frac{[\sin(2\beta_2 - \varphi_{D2}) + \sin \varphi_{D2}] \sigma_3 + 2c_{D2} \cos \varphi_{D2}}{\sin(2\beta_2 - \varphi_{D2}) - \sin \varphi_{D2}} \right\}.
\]

\[
\sigma_1 = \left\{ \frac{(1 + \sin \varphi_{IR}) \sigma_{IR} + 2c_{IR} \cos \varphi_{IR}}{1 - \sin \varphi_{IR}} \right\}.
\]
This relationship is illustrated in Fig. 3.14 for given shear parameters and three different values of \( \sigma_3 \) in the form of a polar diagram. The plot clearly indicates that for the chosen example, the intact rock strength is determining for the strength of the rock mass only when \( \beta_1 = 90^\circ - \beta_2 \) is either approximately 0° or 90°, i.e. when the discontinuities are practically not subjected to shear stresses. With \( c_{IR} = 2 \text{ MN/m}^2 \), a relatively low value was selected for the intact rock's cohesion in the considered example (Fig. 3.14). At higher intact rock's strength, the reduction in the strength of a rock mass due to discontinuities is even more pronounced. The example clearly shows how large the influence of discontinuities on the shear strength of a rock mass can be.

3.4 Models for seepage flow in jointed rock

3.4.1 Homogeneous model and Darcy's law

In soil, it can be assumed that the grains are impermeable and that the flow takes place in the voids. Therefore, we are talking about a porous aquifer. In most cases, the compressibilities of the grain skeleton and of the pore-water can be neglected.

The permeability of the majority of the intact rocks is also very low so that it can be assumed that seepage flow takes only place through the discontinuities. Then, we are talking about a jointed aquifer. Describing the seepage flow through jointed rock, the compressibilities of the rock and of the water in the joints can also be neglected in most cases.

Various approaches to describe the hydraulic behavior of discontinuities are presented in literature (Wittke, 1990). Applying the laws for one-dimensional fissure flow to the flow through discontinuities, it can be shown that usually laminar flow predominates. Turbulent flow conditions arise only at comparatively large aperture of discontinuities or high hydraulic gradients. In general, the assumption of laminar flow is accurate enough for practical cases in rock engineering.

To describe the seepage flow through a jointed rock mass with the aid of a homogenous model, the seepage velocity \( v_f \), which is well-known from soil mechanics, is introduced. It is defined as the quotient of the discharge \( Q \) and the cross-sectional area \( A \) perpendicular to the direction of flow. The cross-sectional area com-
prises not only the cross-section of the discontinuities, but also that of the intact rock, which is assumed to be impermeable here. In Fig. 3.16, the seepage velocity for a rock mass with one set of discontinuities with seepage flow in parallel with the discontinuities (Fig. 3.16a), and for a coarse-grained soil (Fig. 3.16b) is defined. According to Darcy's law, in both cases $v_f$ is proportional to the hydraulic gradient $I$. The factor of proportionality $k_d$ or $k_f$, respectively, is known as coefficient of permeability of the discontinuity set or the soil, respectively.

Fig. 3.16: Definition of the seepage velocity (Wittke, 1990): a) Rock mass with one set of discontinuities; b) coarse-grained soil
The hydraulic gradient $I$ is equal to the change of piezometric head $\Delta h$ related to the length of flow $L$ (Fig. 3.16). The piezometric head is defined as the sum of the geodetic height $z$ and the pressure head $p/\gamma_w$: \[ h = z + \frac{p}{\gamma_w}. \] (3.38)

$p$ is the water pressure and $\gamma_w$ is the unit weight of water. The coordinate system is chosen in such a way that the $z$-axis is directed vertically opposite to gravity.

In case of laminar flow through a discontinuity, the velocity of flow averaged over the aperture $2a_i$, which is referred to as mean velocity of flow $\mathbf{v}$, is given by the following relationship (Wittke, 1990): \[ \mathbf{v} = k_d \cdot I. \] (3.39)

The factor of proportionality $k_d$ is known as coefficient of permeability of the discontinuity. The discharge per meter flowing through $n$ discontinuities ($n = 3$ in Fig. 3.16a) then is \[ Q = k_d \cdot I \cdot n \cdot 2a_i. \] (3.40)

Since the intact rock has been assumed to be impermeable, $Q$ corresponds to the discharge flowing through the entire cross-section \[ A = n \cdot 2a_i + n \cdot (d - 2a_i) = n \cdot d. \]

Thus, the seepage velocity is given by \[ v_c = \frac{Q}{A} = \frac{k_d \cdot I \cdot n \cdot 2a_i}{n \cdot d} = \frac{k_d \cdot 2a_i}{d} \cdot I. \] (3.41)

The coefficient of permeability $k_n$ of the discontinuity set illustrated in Fig. 3.16a then results to: \[ k_n = k_d \cdot \frac{2a_i}{d}. \] (3.42)
If the aperture $2a_i$ is not equal for all discontinuities and the spacing of the discontinuities $d$ is not constant, $2a_i$ and $d$ can be replaced in (3.41) and (3.42) by the mean values $\overline{2a_i}$ and $\overline{d}$.

The relative roughness of the discontinuities' walls is defined as the quotient of the absolute wall roughness $k$ and the hydraulic diameter $D_h$ (Fig. 3.17). In case of laminar flow, the following relationship can be stated for $k_d$ as a function of the relative roughness $k/D_h$ (Wittke, 1990):

$$
  k_d = \begin{cases} 
    \frac{g \cdot (2a_i)^2}{12 \cdot v} & \text{for } k/D_h \leq 0.032 \\
    \frac{g \cdot (2a_i)^2}{12 \cdot v} \cdot \frac{1}{1 + 8.8 \cdot (k/D_h)^{1.5}} & \text{for } k/D_h > 0.032 
  \end{cases}
$$

Fig. 3.17: Definition of relative roughness of a discontinuity (Wittke, 1990)

In equation (3.43), $g$ is the acceleration due to gravity and $v$ the kinematic viscosity of water.

In order to illustrate the magnitude of the permeability of jointed rock, the coefficients of permeability $k_D$ in parallel with a set of discontinuities with an average spacing of $d = 1$ m and differing aperture and roughness are compared with the isotropic coefficients of permeability of certain soils in Fig. 3.18. The coefficient $k_D$ is calculated according to equation (3.42) under consideration of (3.43). As an example, the permeability of a rock mass with a mean aperture of $\overline{2a_i} = 0.2$ mm per meter corresponds to that of a fine sand. With $\overline{2a_i} = 1.0$ mm per meter the permeability of a gravel is obtained.
Equation (3.41) is valid in case the hydraulic gradient $I$ runs parallel with the discontinuities as it is the case in Fig. 3.16a. Generally, the vector of the gradient $\{I\}$, however, also comprises a component normal to the considered set of the discontinuities. Since the intact rock is assumed to be impermeable, this component does not result in a seepage flow. Only the projection $\{I_D\}$ of the hydraulic gradient $\{I\}$ onto the discontinuity plane leads to a seepage flow, which usually is two-dimensional. This dependence on direction can be accounted for with the following equation in a discontinuity-related coordinate system $(x', y', z')$, in which the $z'$-axis is oriented perpendicularly to the discontinuity's plane (Fig. 3.16a; Wittke, 1990):

$$\begin{align*}
\begin{bmatrix}
v_{fDx'} \\
v_{fDy'} \\
v_{fDz'}
\end{bmatrix} &=
\begin{bmatrix}
k_x & 0 & 0 \\
0 & k_y & 0 \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
I_{x'} \\
I_{y'} \\
I_{z'}
\end{bmatrix} \\
&= \begin{bmatrix}
\mathbf{k}_D \\
\mathbf{k}_D \\
\mathbf{k}_D
\end{bmatrix} \\
&= \begin{bmatrix}
k_{D_h} & 0 & 0 \\
0 & k_{D_h} & 0 \\
0 & 0 & k_{D_h}
\end{bmatrix}
\begin{bmatrix}
I_{x'} \\
I_{y'} \\
I_{z'}
\end{bmatrix}
\end{align*}$$

(3.44)

Relating the seepage flow to a fixed global system of coordinates $(x, y, z)$, a transformation of equation (3.44) is necessary. If a coordinate system is chosen in which the $z$-axis is directed vertically, opposite to gravity, and the $y$-axis is directed northwards...
(see Fig. 3.9), then equation (3.44) can be written as (Wittke, 1990):

\[ \{v_{zD}\} = [K_0] \cdot \{I\} \]

(3.45a)

with

\[ [K_0] = k_0 \cdot \begin{bmatrix} 1 - \cos^2 \alpha \cdot \sin^2 \beta & \cos \alpha \cdot \sin \alpha \cdot \sin \beta & -\cos \alpha \cdot \cos \beta \cdot \sin \beta \\ \cos \alpha \cdot \sin \alpha \cdot \sin \beta & 1 - \sin^2 \alpha \cdot \cos^2 \beta & \sin \alpha \cdot \cos \beta \cdot \sin \beta \\ -\cos \alpha \cdot \cos \beta \cdot \sin \beta & \sin \alpha \cdot \cos \beta \cdot \sin \beta & \sin^2 \beta \end{bmatrix} \]  

(3.45b)

Here, the coefficient of permeability \( k_0 \) is replaced by the permeability tensor \([K_0]\) of the set of discontinuities. The angles \( \alpha \) and \( \beta \) are the angles of strike and dip of the set of discontinuities as defined in Fig. 3.4 and Fig. 3.9, respectively.

In the case of several sets of discontinuities \( D_1 \) to \( D_m \), the permeability tensor \([K_f]\) representing the permeability of the entire rock mass can be approximately computed by superimposing the permeability tensors \([K_{0i}]\) of the individual sets of discontinuities (Wittke, 1990):

\[ [K_f] = \sum_{i=1}^{m} [K_{0i}] \]

(3.46)

This is exemplarily shown for an orthogonal system of discontinuities in Fig. 3.19. The energy losses arising at discontinuity intersections, when flows from individual discontinuities meet and when the direction of flow or the cross-section is changed, are neglected. Then, the vector of seepage velocity is obtained from:

\[ \{v\} = [K_f] \cdot \{I\} \]

(3.47a)

with

\[ [K_f] = \begin{bmatrix} k_{fxx} & k_{fxy} & k_{fzz} \\ k_{fxy} & k_{fyy} & k_{fyz} \\ k_{fzz} & k_{fyz} & k_{fzz} \end{bmatrix} \]  

(3.47b)

\[ \{v\} = (v_{fx}, v_{fy}, v_{fz})^T \]

(3.47c)

\[ \{I\} = (I_x, I_y, I_z)^T = \left( \frac{\partial h}{\partial x}, \frac{\partial h}{\partial y}, \frac{\partial h}{\partial z} \right)^T \]  

(3.47d)
Equation (3.47) represents the generalization of Darcy's law for a ground with anisotropic permeability.

If the principal directions of permeability 1, 2 and 3 coincide with the coordinate axes $x$, $y$ and $z$ and in the case of isotropic permeability ($k_f = k_{f1} = k_{f2} = k_{f3}$), $[K_f]$ is a diagonal matrix:

$$[K_f] = \begin{bmatrix} k_{fxx} & 0 & 0 \\ 0 & k_{fyy} & 0 \\ 0 & 0 & k_{fzz} \end{bmatrix} = \begin{bmatrix} k_{f1} & 0 & 0 \\ 0 & k_{f2} & 0 \\ 0 & 0 & k_{f3} \end{bmatrix}$$ (3.47e)
In certain cases, as e.g. for a porous sandstone, the permeability of the intact rock also has to be considered. For steady state flow in such cases the permeability tensor of the entire rock mass can be obtained by superimposing the permeability tensors of the individual sets of discontinuities with a permeability tensor \([K_{IR}]\) describing the isotropic permeability of the intact rock:

\[
[K_{IR}] = \begin{bmatrix}
k_{IR} & 0 & 0 \\
0 & k_{IR} & 0 \\
0 & 0 & k_{IR}
\end{bmatrix}
\]  

(3.48)

Such a superposition is of course also possible if the intact rock's permeability is anisotropic.

During transient seepage flow, the hydraulic gradients in the voids of the intact rock and in the discontinuities generally are different. Using a homogeneous model, transient seepage flow can be described with the concept of double porosity, which can be traced back to Barenblatt et al. (1960). A more recent, short description of this concept can be found e.g. in Wittke (2000).

Equation (3.47) applies not only for jointed rock, but also for soil, for which the permeability particularly in case of coarse-grained soils usually is isotropic (Fig. 3.16b).

### 3.4.2 Equation of seepage flow

The differential equation to describe seepage flow in porous and jointed media can be derived on the basis of the law of mass conservation. According to this conservation principle, the difference of a mass of water flowing into and out of a unit volume per time equals to the change of mass in the unit volume.

The mass of water flowing out of a volume element \(dV\) per time unit \(\dot{m}_w\) (mass flow) amounts to:

\[
\dot{m}_w = \frac{dm_w}{dt} = \rho_w \cdot \frac{dV_w}{dt} = \rho_w \cdot Q = \rho_w \cdot v_f \cdot A
\]  

(3.49)

with

- \(\rho_w\) : density of water,
- \(Q\) : discharge (volume flow),
- \(v_f\) : seepage flow,
- \(A\) : cross-sectional area.
If an infinitesimal small volume element with the edge lengths $dx$, $dy$ and $dz$ is considered, the change of mass flow in $x$-direction is obtained as:

$$dm_{w,x} = \frac{\partial}{\partial x} (\rho_w \cdot v_{t,x}) \cdot dx \cdot dy \cdot dz$$  \hspace{1cm} (3.50a)$$

The mass flow balance in all three coordinate directions yields:

$$dm_w = \left[ \frac{\partial}{\partial x} (\rho_w \cdot v_{t,x}) + \frac{\partial}{\partial y} (\rho_w \cdot v_{t,y}) + \frac{\partial}{\partial z} (\rho_w \cdot v_{t,z}) \right] \cdot dV$$  \hspace{1cm} (3.50b)$$

The mass flow increment $dm_w$ needs to be equal to the change of mass in the element during the time interval considered. The change of mass in the element corresponds to the change of the product of the water density $\rho_w$ and the volume $dV_w$ filled with water (voids or joints):

$$dm_w = -\frac{\partial}{\partial t} (\rho_w \cdot dV_w) = -\frac{\partial}{\partial t} (\rho_w \cdot n) \cdot dV$$  \hspace{1cm} (3.51)$$

with $dV = dx \cdot dy \cdot dz$.

In (3.51), $n$ is the (water saturated) portion of the porosity or relative joint volume, respectively. By equating (3.50) and (3.51), the continuity equation is obtained:

$$\frac{\partial}{\partial x} (\rho_w \cdot v_{t,x}) + \frac{\partial}{\partial y} (\rho_w \cdot v_{t,y}) + \frac{\partial}{\partial z} (\rho_w \cdot v_{t,z}) = -\frac{\partial}{\partial t} (\rho_w \cdot n)$$  \hspace{1cm} (3.52)$$

In most cases, the density of the groundwater can be assumed to be constant (condition of incompressibility). Equation (3.52) can then be simplified as follows:

$$\frac{\partial v_{t,x}}{\partial x} + \frac{\partial v_{t,y}}{\partial y} + \frac{\partial v_{t,z}}{\partial z} = -\frac{\partial n}{\partial t}$$  \hspace{1cm} (3.53)$$

For a steady state seepage flow, the water saturated portion of the porosity or relative joint volume $n$, respectively, is constant, because the piezometric heads are constant with time:

$$\frac{\partial v_{t,x}}{\partial x} + \frac{\partial v_{t,y}}{\partial y} + \frac{\partial v_{t,z}}{\partial z} = 0.$$  

(3.54)
For transient water flow with a free or phreatic surface, the height of the groundwater table is changing. This change, e.g., the groundwater lowering due to the flow towards a tunnel, is described by a specific storage coefficient $S_0$. This coefficient is defined as the volume of water, which can flow out of or into a unit volume of ground with a porosity or relative joint volume $n$, respectively, when the piezometric head is lowered or raised by one unit:

$$ S_0 = \frac{1}{V} \frac{\partial V_w}{\partial h} = \frac{\partial n}{\partial h} \quad (3.55) $$

with $V_w = n \cdot V$. \( (3.56) \)

Inserting (3.55) into (3.53), yields the following continuity equation for incompressible fluids:

$$ \frac{\partial v_{x,t}}{\partial x} + \frac{\partial v_{y,t}}{\partial y} + \frac{\partial v_{z,t}}{\partial z} = -S_0 \frac{\partial h}{\partial t} \quad (3.57) $$

Neglecting the compressibility of the grain skeleton, the water saturated part of the porosity of the ground below the temporary groundwater table is constant. Thus, equation (3.55) only applies for layers in which only part of the total void volume or joint volume $V_p$, respectively, is filled with water so that:

$$ V_w < V_p \quad (3.58) $$

Transient seepage flow in a ground with a free groundwater table, therefore, results mainly from the lowering or raise of the groundwater table.

This is different for a transient seepage flow in a ground with a confined groundwater table. Here, the volume of the water saturated voids can only change due to a local expansion or compression of the latter resulting from an increase or lowering of the piezometric head and thus producing a transient flow. This effect requires the consideration of the deformability of the grain skeleton. In the case of a confined groundwater table, the specific storage coefficients defined according to (3.55) are proportional to the compressibility of the grain skeleton and are orders of magnitude smaller than in the case of a free groundwater table.
In the case of a rock mass with low permeability (e.g. granite or rock salt), the compressibility of the rock mass may have the same order of magnitude as the compressibility of water. Here, not only the compressibility of the rock mass, but also the compressibility of water needs to be taken into account when describing the transient seepage flow in the case of a confined groundwater table. The general form of the continuity equation (3.52) is then to be used. Applications can for example be found in the literature on in situ permeability tests on rocks with low permeability (see e.g. Stormont et al., 1991; Wittke, 1999).

In order to describe the seepage flow in soil and jointed rock, in which flow takes place only in the voids and discontinuities, Darcy's law according to equation (3.47) is inserted into the equation of continuity (3.57). This leads to a partial differential equation of the 2nd order for the piezometric head:

\[
\frac{\partial}{\partial x}\left(k_{t,xx} \cdot \frac{\partial h}{\partial x} + k_{t,xy} \cdot \frac{\partial h}{\partial y} + k_{t,xz} \cdot \frac{\partial h}{\partial z}\right) + \frac{\partial}{\partial y}\left(k_{t,yx} \cdot \frac{\partial h}{\partial x} + k_{t,yy} \cdot \frac{\partial h}{\partial y} + k_{t,yz} \cdot \frac{\partial h}{\partial z}\right) + \frac{\partial}{\partial z}\left(k_{t,xz} \cdot \frac{\partial h}{\partial x} + k_{t,yy} \cdot \frac{\partial h}{\partial y} + k_{t,zz} \cdot \frac{\partial h}{\partial z}\right) = S_0 \cdot \frac{\partial h}{\partial t}.
\]  

(3.59a)

Introducing the Nabla operator

\[
\{\nabla\}^T = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)
\]

(3.60)

and considering equation (3.47b), yields the following abbreviated expression for equation (3.59a):

\[
\{\nabla\}^T \cdot ([K_t] \cdot \{\nabla\} \cdot h) = S_0 \cdot \frac{\partial h}{\partial t}.
\]

(3.59b)

The partial differential equation to describe steady state seepage flow is obtained from (3.59) with \(\frac{\partial h}{\partial t} = 0\):

\[
\{\nabla\}^T \cdot ([K_t] \cdot \{\nabla\} \cdot h) = 0.
\]

(3.61a)
If the permeability of the ground is homogenous, the following applies:

\[ [K_f] \cdot \{\nabla\}^T \{\nabla\} \cdot h = 0. \]  \hspace{1cm} (3.61b)

If the permeability of the ground is not only homogenous but also isotropic, the distribution of piezometric heads is independent of the permeability:

\[ \{\nabla\}^T \{\nabla\} h = 0. \]  \hspace{1cm} (3.61c)

Deriving the continuity equation, a closed system was assumed for the volume element considered. Thus, water added to or extracted from the element, e.g. springs or depressions, is not accounted for. Adding appropriate quantities of water \(Q_v\), which flow into or out of a unit volume per time unit, to the left hand side of equation (3.59b) or (3.61a), respectively, yields the following differential equations describing the transient and steady state seepage flow:

\[ \{\nabla\}^T \left( [K_f] \cdot \{\nabla\} \cdot h \right) + Q_v = S_0 \frac{\partial h}{\partial t}, \]  \hspace{1cm} (3.59c)

\[ \{\nabla\}^T \left( [K_f] \cdot \{\nabla\} \cdot h \right) + Q_v = 0. \]  \hspace{1cm} (3.61d)

### 3.4.3 Hydrostatic uplift and seepage force

For stability analyses, the forces transmitted from the water to the ground and to the tunnel lining must be known.

In the area below the groundwater table, the grain skeleton of a soil and the blocks of intact rock surrounded by discontinuities of a rock mass, respectively, are subjected to hydrostatic uplift such as a solid body in water. The hydrostatic uplift is a body force, which is directed opposite to gravity. According to Archimedes' law, the uplift is equal to the weight of the displaced water volume. Therefore, the unit weight of a soil subjected to hydrostatic uplift amounts to:

\[ \gamma' = \gamma_d - \gamma_w \cdot (1 - n). \]  \hspace{1cm} (3.62)
In equation (3.62), \( \gamma_d \) in the unit weight of the dry soil, \( \gamma_w \) is the unit weight of water and \( n \) is the soil's porosity. The uplift force \( U \) is, according to Archimedes' law, equal to the weight of the volume of water displaced by the soil grains:

\[
U = \gamma_w \cdot (1 - n).
\]  
(3.63)

The definition of uplift force according to (3.63) is related to the solid body, i.e., to the individual grains of the soil. In soil mechanics, the uplift force is normally related to the total volume of soil, composed of grains and water filled voids. Then, the unit weight of dry soil \( \gamma_d \) in equation (3.62) is replaced by the unit weight of saturated soil \( \gamma_{sat} \). Inserting

\[
\gamma_d = \gamma_{sat} - n \cdot \gamma_w
\]  
(3.64)

in (3.62) yields

\[
\gamma' = \gamma_{sat} - \gamma_w
\]  
(3.65)

with \( F_U = \gamma_w \)  
(3.66)

as the uplift force which is commonly used in soil mechanics.

In jointed rock, the porosity \( n \) is replaced by the relative joint volume \( n_j \), which is related to the total volume. The joint volume normally is small enough to be neglected in comparison with the total volume. Thus, the following applies for the unit weight of a rock mass which is subjected to uplift:

\[
\gamma' = \gamma_{sat} - \gamma_w \approx \gamma_d - \gamma_w.
\]  
(3.67)

In a coordinate system, in which the z-axis is oriented opposite to gravity, the uplift force \( F_U \) according to (3.66) can be expressed in the form of a vector as follows (Wittke, 1990):

\[
\{F_u\} = \gamma_w \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.
\]  
(3.68)
If the groundwater table is lowered during the excavation of a tunnel or in case of a drained tunnel, a seepage flow will be initiated in the soil or in the rock mass respectively, and the soil's grain skeleton or the intact rock are additionally loaded by a hydrodynamic force. This force, which acts on the grain skeleton or the intact rock, is also a body force and is known as seepage force $F_S$. Related to a unit volume of the soil or the rock mass, this force in vector representation can be written as:

$$\{F_S\} = \gamma_w \cdot \{I\}.$$  \hspace{1cm} (3.69)

The deviation of (3.69) can, e.g., be found in Wittke (1990). According to (3.69), the direction of the seepage force $\{F_S\}$ coincides with the direction of the hydraulic gradient $\{I\}$.

### 3.5 Analysis methods

#### 3.5.1 Finite element method (FEM)

Analysis procedures to investigate the stability of tunnels must account for the models for stress-strain behavior and seepage flow described in sections 3.3 and 3.4, for inhomogeneous ground conditions as well as for arbitrary geometry of the excavated cross-section. Furthermore, the computation of general three-dimensional states of stress and deformation, which occur in the area of the temporary tunnel face, the computation of uplift and seepage forces as well as the simulation of support measures and different stages and sequences of construction are required.

The finite element method (FEM) has proven to be especially powerful and in recent years has come to a wide application in the field of geotechnical engineering, particularly in tunneling. The FEM as well as its application to the models described in Sections 3.3 and 3.4 are treated in detail e.g. in Wittke (2000).

The FEM is based on the subdivision of the selected continuum's section of analysis into individual elements of finite size (Fig. 3.20), the so-called finite elements. The finite elements are connected with each other by a finite number of nodal points located on the element's boundaries.
Fig. 3.20: Discretization of the section of the continuum selected for stability analyses (computation section) in case of a machine-driven tunnel with segmental lining (Wittke-Schmitt and Lorenzo Martin, 2006)
In the case of elastic behavior, a linear relationship between the nodal point displacements and the nodal forces resulting from volume forces (e.g. self-weight), external loads and displacements applied on the boundaries, can be derived using e.g. the principle of minimizing the potential energy (Wittke, 1990). The computation of displacements as well as strains and stresses derived from these can be put down to the solution of a system of linear equations. Under consideration of viscoplastic strains, the nodal point displacements can be calculated with the aid of the time-history analysis described in Section 3.3 by repeatedly solving the system of equations with modified nodal forces (Wittke, 2000).

The computation of the piezometric heads in the nodal points and of the uplift and seepage forces derived from these can in the same manner be put down to the solution of a system of linear equations (Wittke, 2000).

**Computation section**

Since tunnels are linear structures, stability analyses in many cases can be carried out by way of approximation, using vertical rock mass slices. When the ground is isotropic and the boundary conditions are chosen adequately, the analyses may be restricted to plane strain conditions and are therefore two-dimensional. In the case of anisotropic stress-strain behavior, the strains in parallel with the tunnel's axis are also equal to zero, however, displacements can occur. Thus, the state of deformation may be three-dimensional. This state of deformation, which is also referred to as generalized plane strain (Lekhnitskii, 1963), can also be considered using a vertical rock mass slice if the boundary conditions are chosen appropriately. Such an analysis is also referred to as pseudo-threedimensional (Wittke, 2000).

In determining the size and form of the computation section, care must be taken to ensure that displacements due to excavation do not lead to constraints at the computation section's boundaries, causing additional stresses which falsify the computation result. Consequently, the computation section has to be chosen large enough so that stress changes and deformations due to excavation do not reach the section's boundary. This requirement has to be fulfilled in any case. When defining the dimensions of the computation section, various influences must be considered, which will exemplarily be dealt with in the following.
The dimensions and the shape of the tunnel's cross-section are decisive for the required size of the computation section. For a tunnel in an isotropic elastic ground with a horizontal ground surface and a predominantly vertical loading, Fig. 3.21 gives, as an example, an indication for the dependence of the required dimensions of the computation section on the dimensions of the tunnel's cross-section, which is assumed to be circular. Since the ground conditions as well as the tunnel's geometry are symmetric, the computation section can be reduced to half the size required for the given tunnel's dimension in this case.

Fig. 3.21: Dimensions of the computation section, isotropic elastic ground, horizontal ground surface, loading due to self-weight (Wittke, 2000)
Fig. 3.22: Influence on the required size of the computation section (Wittke, 2000): a) Isotropic elastic ground, loaded by self-weight; b) isotropic elastic ground, high horizontal stresses; c) anisotropic ground, formation of plastic zones; d) inhomogeneities

The state of stress existing in the ground before excavation of the tunnel results from the loading of the ground in undisturbed state and is referred to as primary state of stress \( \{\sigma_p\} \). If this state of stress results only from the self-weight of the rock mass, the maximum principal normal stress is oriented vertically. In this case, the extent of the rock mass areas above and below the tunnel which are unloaded due to the excavation is larger than the additionally loaded areas of the rock mass adjacent to the
tunnel. Thus, the height of the slice determines the required dimensions of the computation section (Fig. 3.21 and Fig. 3.22a). If the horizontal stress in the ground is higher than the vertical stress resulting from self-weight, the situation is reversed so that the required width of the computation section becomes decisive (Fig. 3.22b). The larger the ratio of horizontal to vertical stress is, the larger the computation section's width must be.

When the stress-strain behavior of the ground is anisotropic, the stress changes and the deformations resulting from tunneling may be unsymmetric with respect to a vertical plane running through the axis of the tunnel. In this case, the computation section must include the entire cross-section of the tunnel and the distance between the tunnel's walls and the boundaries of the computation section has to be large enough on both sides of the tunnel (Fig. 3.22c).

A formation of plastic zones adjacent to the tunnel can also have a significant influence on the size of the computation section, because a stress redistribution via this zones is not possible or at least limited. Plastic zones in the ground, as represented in Fig. 2.22c, therefore have the same statical effect as a widening of the cross-section and thus require a correspondingly greater width of the computation section.

Inhomogeneities in the ground, such as a steeply dipping fault (Fig. 3.22d), can lead to stress redistributions and displacements in areas far away from the tunnel. The computation section must therefore be chosen appropriately large. A seepage flow, eventually caused by the excavation of the tunnel, results in seepage and uplift forces which lead to stress changes and displacements that may reach to areas far away from the tunnel. Thus, considering the influence of seepage flow in such cases, a larger computation section often has to be chosen.

The dimensions of the computation section have to be determined for each case under consideration of the above mentioned criteria, in which experience in applying the FEM is of great value. In general, it should be verified on the basis of the computation results whether the computation section has been chosen large enough.
Discretization

Besides the size of the computation section, its discretization into finite elements has an influence on the accuracy of the calculated displacements and stresses. The finer an element mesh at a given number of nodal points per element, the more accurately the calculated displacements and stresses correspond to the exact solution. The accuracy of the computation results in particular can be raised by an increase of the number of nodal points per element. The utilization of isoparametric three-dimensional elements with 8 to 21 nodal points, which are described in detail e.g. in Zienkiewicz (1977), Wittke (1990) and Wittke (2000) is recommended.

When an elastic rock mass is considered, the displacements of the nodal points normally can be computed rather exactly even with relatively coarsely discretized meshes consisting of 8 node elements. The same accuracy regarding the calculation of stresses, however, requires finer meshes. Basically, the element mesh should be particularly fine in all those areas where large stress changes occur. For tunnels, this is always the case near the opening, because the stress changes decrease rapidly with increasing distance from the opening.

WBI nowadays predominantly uses elements with 16 to 20 nodal points, because of the higher accuracy of the analysis results. The order of the interpolation functions for the displacements in these elements is one degree higher than for elements with 8 nodal points. Consequently, with the same refinement of meshes, the accuracy of the calculated displacements and stresses is higher. For three-dimensional analyses, mainly elements with 20 nodal points and with intermediate nodal points at all edges are used. Carrying out two- or pseudo-three-dimensional analyses where displacements and stresses in parallel with the tunnel's axis do not change, the intermediate nodal points are left out on the element edges oriented in parallel with the tunnel's axis. Thus, the elements used for the simulation of the ground and the lining have 16 nodal points each (see Fig. 3.20).

Boundary conditions

When defining the boundary conditions, the forces and displacements which are imposed to the boundaries of the computation sec-
tion have to be chosen in such a manner that the primary state of stress \( \{\sigma_p\} \) existing in the undisturbed ground is correctly simulated.

In the case of isotropic behavior of the ground, as represented for the case in Fig. 3.23, the nodal points of the element mesh lying on the lateral and lower boundaries of the computation section, respectively \((x = 0 \text{ and } z = 0)\) can be assumed to be fixed in the direction perpendicular to their respective boundary. Corresponding boundary conditions can also be assumed for the nodal points located on the planes \(x = L/2, y = 0\) and \(y = D\). The plane \(x = L/2\) represents a plane of symmetry.

Fig. 3.23: Displacement and force boundary conditions for isotropic ground (Wittke, 2000)

In case of seepage flow, the uplift and seepage forces are determined by means of a seepage flow analysis and then applied as
equivalent nodal forces, as it is done for the weight of overburden at the upper boundary of the computation section and for anchor forces (Fig. 3.23).

Fig. 3.24: Displacement boundary conditions for an anisotropic rock mass, pseudo-three-dimensional analysis (Wittke, 2000)

planes \( x = 0 \) and \( x = L \):
equal displacements for nodal points having the same \( y \) and \( z \) coordinates, e.g. \( \{ \delta_0 \} = \{ \delta_m \} \) and \( \{ \delta_p \} = \{ \delta_q \} \)

planes \( y = 0 \) and \( y = D \):
equal displacements for nodal points having the same \( x \) and \( z \) coordinates, e.g. \( \{ \delta_l \} = \{ \delta_m \} \)
Since in an anisotropic rock mass even self-weight can cause displacement components in horizontal direction, it is in general not allowed to fix the nodal points lying on the computation section's vertical boundary planes, in the directions perpendicular to the latter. In the case represented in Fig. 3.24, the same displacements are assigned to the nodal points on the planes $y = 0$ and $y = D$, the location of which only differs in the $y$-coordinate (e. g. $\{\delta_1\} = \{\delta_n\}$). Also to the nodal points on the lateral planes $x = 0$ and $x = L$ having the same $y$ and $z$ coordinates, e. g. points 0 and $n^*$ as well as $p$ and $q^*$, the same displacements are assigned ($\{\delta_o\} = \{\delta_{n^*}\}$, $\{\delta_p\} = \{\delta_{q^*}\}$).

Fig. 3.25: Simulation of a horizontally oriented tectonic normal stress in an isotropic elastic ground loaded by self-weight (Wittke, 2000)
Increased horizontal stresses in the ground, e. g. due to tectonic loading or diagenetic hardening, as represented in Fig. 3.25, can be simulated by imposing a displacement $\delta_x$ on the nodal points lying on the boundary $x = 0$, which leads to the requested stress. The relationship stated in Fig. 3.25 for the calculation of $\delta_x$ can be derived with the aid of the generalized Hook's law (see Section 3.3).

The equations for seepage flow, derived in Section 3.4, can also be solved with the aid of the FEM for any boundary conditions.

The boundary conditions for the seepage flow towards a tunnel in a homogeneous isotropic ground are illustrated in Fig. 3.26. Due to symmetry, only half of the system is considered in the computation section.

Fig. 3.26: Boundary conditions for seepage flow analyses, tunnel in a homogeneous isotropic ground (Wittke, 2000)
The specification of constant or variable volumes of water per unit of time (discharges $Q$) at one or more boundaries of the computation section is known as Neumann boundary condition. Such boundary conditions are introduced as springs or depressions. Neumann boundary conditions which are variable with time are called flow hydrographs. Flow hydrographs can for example be applied on the surface of the computation section (plane 1 in Fig. 3.26) as variable discharges due to precipitation. A special case of Neumann boundary conditions are impermeable boundaries which are specified as streamlines. At the streamlines the discharges are equal to zero (planes 2 and 3 in Fig. 3.26).

The specification of constant or variable piezometric heads at one or more boundaries of the computation section is known as Dirichlet boundary condition. Time dependent Dirichlet boundary conditions are referred to as piezometric head hydrographs. These can for example be specified at boundaries with known water level variations (plane 4 in Fig. 3.26). In this case, all nodes located on plane 4 have the same piezometric head. Plane 4 thus represents an equipotential surface. The piezometric heads at the external boundaries (plane 4) are often specified as piezometric heads corresponding to the undisturbed water level. In such cases, however, the computation section must be selected large enough so that the distance of the boundary to the tunnel is equal to or larger than the expected range of the groundwater lowering.

Special cases of the Dirichlet boundary conditions are the free or phreatic surface (plane 5 in Fig. 3.26) and the seepage surface (plane 6 in Fig. 3.26). The piezometric head of all nodes lying on the phreatic and the seepage surface is equal to the point's geodetic height ($h = z$). The extent of the seepage surface as well as the location of the phreatic surface usually are unknown at the beginning of the analysis and must be determined iteratively (Wittke, 2000).

The effect of drainage holes can be simulated by specifying the respective piezometric head at individual nodes. The piezometric heads for a fully effective drainage with drainholes inclined below the horizontal are equal to the geodetic height at the heads of the corresponding boreholes. At the corresponding nodes, water can leave the computation section at a rate which can be calculated from the computed piezometric heads.
The effect of wells with a constant pumping capacity can be simulated by specifying discharges at the corresponding nodes.

Furthermore, it can be specified as boundary conditions that the piezometric heads at two nodes are initially unknown, but equal to each other (Wittke, 1990). Introducing such boundary conditions, the computation effort can be reduced, because then a three-dimensional analysis can, under certain circumstances, be replaced by a two-dimensional (pseudo-three-dimensional) analysis.

**Analysis of uplift forces, seepage forces and water pressure**

The loading of the rock mass due to seepage flow can be simulated by nodal forces. The conversion of the uplift and seepage forces, introduced in Section 3.4.3, into nodal forces is carried out elementwise. If part of the element is located above the phreatic surface, for which \( h < z \) applies, the element is only partially subjected to a seepage force resulting from flow (Wittke, 2000).

In the case of impermeable tunnel linings, a hydrostatic water pressure on the lining has to be considered which can be derived from the piezometric head and the geodetic height:

\[
p_w = \gamma_w \cdot (h - z)
\]  

(3.70)

This water pressure can be applied to the tunnel lining with the aid of equivalent nodal forces.

In the case of a sealed tunnel with a double lining, a sealing is installed between the external lining consisting of shotcrete and the internal lining consisting of reinforced concrete. To limit the shear stresses between the internal lining and the external shotcrete membrane, the sealing is separated from the shotcrete membrane by fleece. Since the shotcrete membrane as well as the fleece are permeable compared to the internal lining, the water pressure prevailing in the rock mass can build-up within the fleece along its complete circumference (\( p_w \) in Fig. 3.27). This water pressure \( p_w \) needs to be applied on the internal lining. The rock mass below the groundwater table is subjected to hydrostatic uplift.
Fig. 3.27: Reinforced concrete lining with outside synthetic sealing and permeable fleece. Assumption for water pressure distribution in jointed rock (Wittke et al., 2004)

In case of a machine-driven tunnel with a subsequent single segmental lining, the situation is different. Depending on potential support measures at the temporary face, at least a partial lowering of the groundwater table may occur. But after installation of the segmental lining it rises to its original level again.
Since the annular gap between the rock mass and the segmental lining is grouted with mortar, no water pressure can act in this gap. But within the discontinuities, which are located next to the excavation contour, a water pressure arises. This water pressure is acting on the segmental lining and according to Fig. 3.28 is increasing with depth (Wittke, et al., 2004; Wittke and Wittke-Gattermann, 2005). Thus, the same water pressure loading as in the case of a sealed reinforced concrete lining results (compare Fig. 3.27 and 3.28).

This approach is only valid if the spacing of the water-filled discontinuities is small compared with the diameter of the tunnel. In most practical cases, this condition is fulfilled (Wittke et al., 2004). In soil, the water pressure is to be applied in the same way.

3.5.2 Simplified approaches

A model, which is frequently used for the dimensioning of the shield's skin and of the segmental lining respectively, is the shell model with elastic bedding of the shield's skin and the segmental lining, which are substituted by truss or shell elements respectively. The bearing structure, shield skin or segmental lining, is considered separately from the ground (Fig. 3.29). Self-weight, rock mass pressure and water pressure are applied as external loads (Duddek, 1980; DAUB, 2005).

The weight of overburden and potential additional loads due to water pressure, traffic and buildings can be applied as vertical loading of the shield and of the segmental lining respectively for shallow tunnels located in soil. The horizontal loading is determined by the coefficient of lateral earth pressure K and a potential additional horizontal stress $\Delta \sigma_h$, e. g. as a result of an overconsolidation of the soil due to preloading. Also the driving of curves or corrections during driving can locally lead to an increased horizontal pressure on the shield's skin. These influences, however, are difficult to evaluate and thus are normally estimated on the basis of experience. K should be selected in such a way that the horizontal load corresponds either to the earth pressure at rest or to the active earth pressure. In each case the more unfavorable value should be chosen.
Fig. 3.28: Segmental lining for a tunnel in jointed rock with annular gap grouted with mortar. Assumption for water pressure distribution (Wittke et al., 2004)
For tunnels with a higher overburden, arching in the rock mass according to the silo theory may be considered, by loading the shield's skin and the segmental lining by a failure body which may even reach to the ground surface.

Fig. 3.29: Application of earth pressure and elastic bedding to the shield's skin and the segmental lining, respectively

The loaded shield skin and the segmental lining respectively can be analysed with and without an elastic bedding. In the bedding approach shown in Fig. 3.29, the roof area with an aperture angle $\alpha$ is not bedded. In the remaining area, the reaction of the ground is modeled by springs (Duddek, 1980). The spring constants, which are referred to as modulus of subgrade reaction $k_r$, are estimated on the basis of experience. As a conservative assumption the tangential bedding is normally not considered.

Single as well as several, but normally two, coupled segmental rings can be modeled using truss or shell elements (Fig. 3.30). The longitudinal joints can be modeled with torsion springs. The points of potential contact within the circumferential joints of coupled rings are simulated by truss members.
In the described model, the interaction between structure and ground is approximated by preset load assumptions and assumptions regarding magnitude, direction and distribution of the modulus of subgrade reaction. Using this model, it is not possible to determine subsidence at the ground surface or at the foundations of existing buildings and stress-strain states in the surrounding ground.

Fig. 3.30: Segmental lining simulated by truss or shell elements with elastic bedding (Balthaus et al., 2005)

For the design of the lining of shield-driven tunnels with small overburden located in soil, the model with elastic bedding can, however, be used, because the segmental lining in such cases normally can be dimensioned for the entire weight of overburden without difficulties. In the case of tunnels with high overburden, however, the arching in the rock mass must be accounted for in the analysis. The model with elastic bedding, therefore, should in such cases only be applied for approximate calculation.